Residual Monte Carlo Methods

Thomas Evans, Scott Mosher, CCS-2

In Ref. [1], we developed a residual Monte Carlo method that was an application of Halton's Sequential Monte Carlo method [2] to the (1-D) equilibrium (1T) thermal radiation diffusion equation. At that time, Halton's method was not widely known among numerical practitioners. While extending our method to 3-D, we have discovered that the Sequential Monte Carlo method is actually a variant of iterative refinement cast as a residual method. Based more rigorously on iterative refinement, our new Monte Carlo method surpasses our previous 1-D method and allows for extension to more general solution techniques.

The goal of recent work is to develop efficient Monte Carlo solvers for discrete systems. We have developed fast linear solvers using Monte Carlo techniques with two parallel approaches in this effort: one approach for general linear systems and the other for nonsymmetric, multidimension, banded systems that result from discretized partial differential equations.

The focus of the second approach is to solve 3-D diffusion equations for nonlinear thermal radiative transfer. Along these lines, we have extended our previous work by developing an Iterative-Refinement Monte Carlo (IRMC) method for solving sparse matrix systems. We applied this solver to the 1T thermal radiation diffusion equation in 3-D Cartesian geometry.

Consider the following preconditioned, discrete linear system,

$$\mathbf{M}^{-1}\mathbf{D}\phi = \mathbf{M}^{-1}\mathbf{q}. \quad (1)$$

The only requirement on the preconditioned system is that the spectral radius of $\mathbf{M}^{-1}\mathbf{D}$ must be less than 1. The IRMC method that solves (1) is defined

$$\boldsymbol{\phi}^{l+1/2} = (\mathbf{I} - \mathbf{M}^{-1}\mathbf{D})\boldsymbol{\phi}^{l} + \mathbf{M}^{-1}\mathbf{q} \ ,$$
(fixed-point iteration)

$$\mathbf{r}^{l+1/2} = \mathbf{M}^{-1}\mathbf{q} - \mathbf{M}^{-1}\mathbf{D}\boldsymbol{\phi}^{l+1/2},$$
(estimate residual)

$$\mathbf{M}^{-1}\mathbf{D}\delta\boldsymbol{\phi}^{l+1/2} = \mathbf{r}^{l+1/2} \ ,$$
(solve using Monte Carlo)

$$\boldsymbol{\phi}^{l+1} = \boldsymbol{\phi}^{l+1/2} + \delta\boldsymbol{\phi}^{l+1/2} \ .$$
(update ϕ)

The update of the correction term is performed using a Monte Carlo transport calculation [3].

We have applied the IRMC method to the 3-D, 1T, nonlinear radiation diffusion equation. A multimaterial duct problem is shown in Fig. 1. This problem features a 0.5 keV blackbody flux on the low-*x* side. Radiation is propagated through a dog-legged duct bounded by an opaque wall. An optically thick foil is placed on the high-*y* side of the outlet. A contour plot of the solution is shown in Fig. 2, and the time-evolution of the temperature at four edit points is shown in Fig. 3. The Monte Carlo solution can be run to arbitrary precision because the convergence of the IRMC method is not bound by the Central Limit Theorem.

We have compared the IRMC method on this problem with standard solution techniques. The IRMC method compared with preconditioned Conjugate Gradient (CG) and Richardson iteration are shown in Table 1. The IRMC method compares very favorably to preconditioned Richardson iteration and is marginally faster than preconditioned CG.

Table 1: Comparison of solution methods for the multimaterial problem. The problem was run to an elapsed time of 10 sh requiring 10157 cycles.

Method	Max Iterations	CPU Time (s)
Preconditioned CG	18	12794.3
Preconditioned Richardson	38	17677.6
IRMC	20	12367.4

We have presented a new Monte Carlo solution method for solving the discrete, time-dependent diffusion equation in 3-D. The IRMC method has been shown to match results using standard solution techniques to arbitrary precision. Also, the new method is faster than preconditioned CG and Richardson iteration.

While we have demonstrated marginal improvements over standard solution schemes in this study, significant improvements could be realized in fully nonlinear-consistent solutions. In these cases, the IRMC method competes with GMRES, which is more costly in memory and time than CG. Another area where the IRMC scheme may have advantages over traditional solution methods is on dentritic, or adaptive, meshes. These meshes yield matrices with poor condition numbers because of the changing cell volumes at different refinement levels. A smart transport algorithm could be developed that more efficiently solves the residual on these types of meshes. These topics will be the focus of future study.

For more information contact Thomas M. Evans at tme@lanl.gov.

[1] T. Evans, et al., *J. Comp. Phys.* **189**, pp. 539–556, 2003.
[2] J. Halton, *J. Sci. Computing* **9**(2), pp. 213–257, 1994.
[3] J. Hammersly and D. Handscomb, *Monte Carlo Methods* (Spottiswoode, Ballantyne, and Co., London, 1964).

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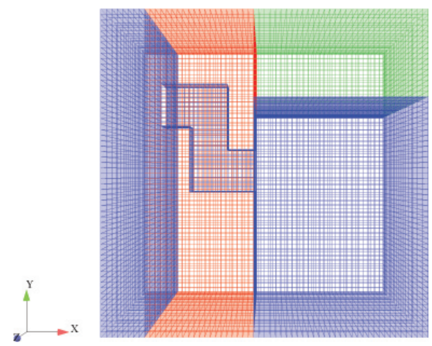

Fig. 1.
Multimaterial problem mesh (60x60x60): blue shows the duct region, red shows the wall, and green shows the foil.

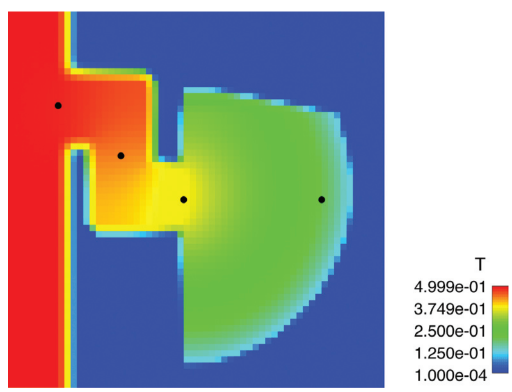

Fig. 2.
Contour plot of the temperature at 100 sh on a cut-plane positioned at the midpoint of the z-axis.

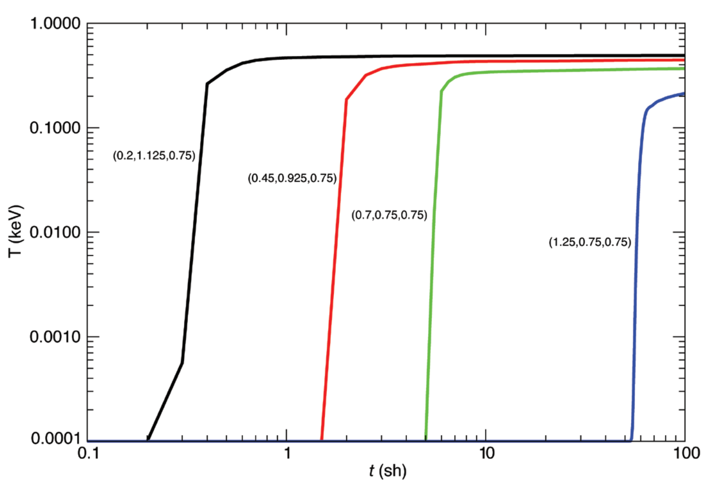

Fig. 3.
Time evolution of the temperature at the edit points shown in Fig. 2.